Hose Stream Model PSMT – Eden Tomes

# Introduction:

Computer-generated imagery in video games relies on mathematical models to simulate real-world phenomena, such as the motion of objects under gravity. This report addresses a task given by a computer animation studio to develop a mathematical model representing the path of a water stream from a hose, through the application of knowledge surrounding quadratic and cubic graphical functions. The goal is to create a function that accurately mimics the trajectory of the water in a given image, serving as a proof of concept for a physics engine.

To achieve this, three successive mathematical models were developed:

1. A manually calculated quadratic function in factored form ()
2. A quadratic regression model ()
3. A cubic regression model ()

This report evaluates the accuracy of each model by comparing their resulting parabola & residuals and discussing strengths and limitations in replicating the observed trajectory. The conclusions drawn from this analysis will determine which function best replicates the observed water trajectory, ultimately informing the development of a more realistic and accurate physics engine for the animation studio.

# Assumptions:

* It is assumed that the hose is positioned at ground level and the water stream begins exactly at the origin (0, 0) on a Cartesian plane. This simplification reduces problems caused by initial height or lateral displacement, allowing for a direct application of mathematical models without overly complicated coordinate adjustments.
* The models assume the water stream is a two-dimensional projectile that is not affected by air resistance, wind, or other external forces. Exclusion of these factors isolates the effect of gravity on the stream’s trajectory, making comparison of the models’ accuracies more straightforward.
* It is assumed that the data points collected through manual estimation will allow accurate translation of the water stream’s physical profile. This assumption has significant implications for the formation and testing of the models, as this data serves as the basis of comparison between models.

# Observations:

* There is considerable interest from a computer animation studio in applying mathematical modeling techniques to real-world physical phenomena, such as the behavior of water streams. This interest underscores the study’s purpose and emphasizes the significance of developing an accurate model for the water stream’s trajectory.
* The water stream in the image follows a distinct, curved path that mirrors typical projectile motion. This supports the use of quadratic and cubic functions as effective tools for modeling a smooth, continuous curve influenced by gravity.
* Although the water stream generally exhibits a consistent parabolic trajectory due to gravity’s predictable influence, subtle deviations can be observed. These result from variations in water pressure, slight turbulence, or imperfections during manual data collection, all of which are important to consider when evaluating model accuracy.

# Method:

### Data Collection

|  |  |
| --- | --- |
|  | To gather the necessary data for the mathematical models, the trajectory of the water stream in the provided image (pictured left) was analyzed. The image was imported into the Desmos online graphing calculator, where the hose nozzle was aligned at the origin (0, 0) and the landing point was set at (10, 0). Intermediate points along the x-axis were recorded at 0.5-unit intervals, with the corresponding y-values visually estimated and rounded to the nearest 0.05.  The data points (see Table 1) were then used to develop three distinct models: a manually derived quadratic function, a quadratic regression model, and a cubic regression model, both generated using Excel. |

## Table 1: Observed coordinates of water stream

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | (Observed value) |  | (Observed value) |  |
| **0.0** | 0.00 | **5.5** | 10.15 |
| **0.5** | 1.75 | **6.0** | 10.00 |
| **1.0** | 3.25 | **6.5** | 9.70 |
| **1.5** | 4.65 | **7.0** | 9.20 |
| **2.0** | 5.85 | **7.5** | 8.40 |
| **2.5** | 6.95 | **8.0** | 7.20 |
| **3.0** | 7.85 | **8.5** | 5.90 |
| **3.5** | 8.65 | **9.0** | 4.20 |
| **4.0** | 9.30 | **9.5** | 2.20 |
| **4.5** | 9.70 | **10.0** | 0 |
| **5.0** | 10.00 |  |  |

### Development Of Models

Three different approaches were used to develop the mathematical models, each refining the fit to the observed data (see Table 1).

* First, a manually derived quadratic model was calculated using the factorized (or x-intercept) form of a quadratic equation.

With the known x-intercepts at 0 and 10, the equation was set up as:

By substituting the mid-point (5, 10) into the equation, the coefficient was determined:  
Let x = 5, y = 10:

Thus, the resulting equation became:

* Next, a quadratic regression model was generated using Excel. The observed data points were graphed, and a polynomial trendline of degree 2 was added. The resulting equation was:
* Finally, a cubic regression model was created by applying a degree 3 polynomial trendline in Excel. This further refined the approximation of the water stream’s trajectory. The derived cubic equation was:

# Results:

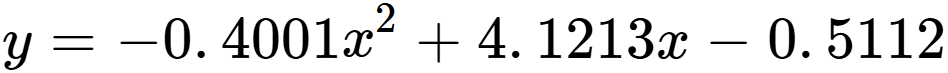
## Table 2: Residuals of mathematical models

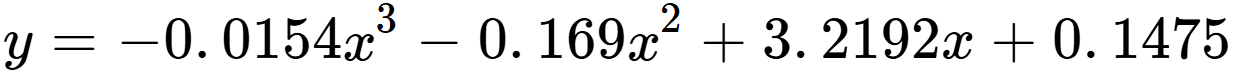
|  |  |  |  |
| --- | --- | --- | --- |
|  | **Solution 1** | **Solution 2** | **Solution 3** |
| **Sum Of Residuals** | 9 | 6.2277 | 1.2143 |
| **Average Residual** | 0.4286 | 0.2966 | 0.0578 |

|  |  |
| --- | --- |
| Graph 1: Line scatterplot of Solution 1 parabola | Graph 2: Line scatterplot of Solution 2 parabola |
|  |  |

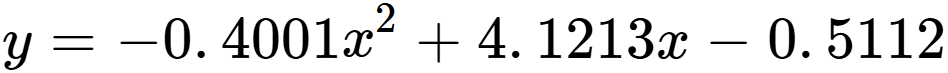
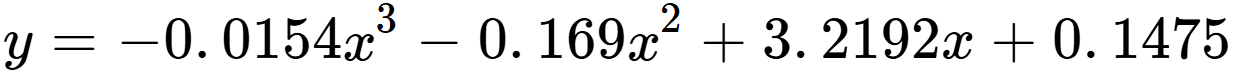
## Graph 3: Line scatterplot of Solution 3 parabola

A visual comparison of the models’ parabolas and their residual values (difference to the observed values) highlights clear differences in their accuracy. The manually derived quadratic function , shown in Graph 1 (green), loosely follows the expected parabolic shape of the water stream, consistent with a basic projectile arc. However, it shows noticeable differences from the observed values (red), particularly around the midpoint (x=5) and the endpoint (x=10). These misalignments are supported by the model’s residual values, with a sum of 9.0 and an average of 0.4286—the highest of the three solutions. The fixed intercepts at (0, 0) and (10, 0) likely reduce the model’s ability to accommodate the small variations in the data, leading to these gaps.

The quadratic regression model (), plotted in Graph 2 (blue), shows a stronger fit to the observed data. By determining the quadratic function through regression, this model better matches the peak height near x=5 and reduces deviations in the mid-region. This improvement is reflected in its lower residuals, with a sum of 6.2277 and an average of 0.2966. While Solution 2 maintains a similar parabolic shape to the first solution, the mathematical precision of regression allows it to better approximate the water stream’s path compared to Solution 1.

The cubic regression model (), shown in Graph 3 (purple), best matches the observed data. The cubic function allows the model to capture slight asymmetries, including the subtle flattening around the peak and the steeper decline after x = 7. The parabola closely follows the observed data points (red), as confirmed by its substantially lower residuals (sum: 1.2143 and average: 0.0578). This accuracy suggests that the cubic model can accommodate small variations (potentially arising from factors like fluctuations in water pressure or minor errors in data estimation) and the asymmetry from the influence of gravity, that a simpler quadratic model would not fully capture.

# Evaluation:

The three mathematical models—manually derived quadratic, quadratic regression, and cubic regression—were assessed to determine their suitability for simulating the water stream’s trajectory. The analysis combined quantitative metrics, such as residual values, with qualitative visual comparisons to evaluate how closely each model aligned with the observed data. The manual quadratic model (), while adhering to the basic principles of projectile motion, showed noticeable deviations from observed values, particularly near the midpoint (x = 5) and endpoint (x = 10). Its residuals, totaling 9.0 with an average of 0.4286, underscored these gaps. The quadratic regression model () demonstrated improved accuracy by reducing the sum of residuals to 6.2277 and the average to 0.2966, a proportional decrease of about 30 percent. The final model () using cubic regression achieved the closest alignment, with residuals as low as 1.2143 (average: 0.0578), nearly overlapping the observed data points.

### Reasonableness of Solution

The assumption of a simplified 2D projectile motion (no air resistance, ground-level origin) allowed all models to focus purely on gravitational effects. The manual quadratic model’s symmetry aligned with this assumption but failed to account for slight asymmetries in the observed data, making it less reasonable for real-world applications. The regression models, particularly the cubic, better accommodated these deviations without violating core assumptions, enhancing their practicality.

The observed stream trajectory of the image does adhere to projectile motion but included minor irregularities, likely from water pressure variations or data estimation errors. While the first two quadratic models captured the general parabolic trend, the cubic regression’s flexibility to model asymmetries matched the observations more closely. This supports its reasonableness as a solution for replicating subtle real-world behaviors of objects under the influence of gravity.

# Strengths of the Study:

* The study’s methodology ensured a balanced evaluation of model accuracy and practicality. By testing multiple approaches (manual calculation, quadratic regression, cubic regression) it highlighted the trade-offs between simplicity and precision.
* Utilization of established mathematical techniques like quadratic/cubic functions and polynomial regression rationalizes the wide scale application of the findings, making it highly suitable as a proof of concept for a physics engine.
* Performance of visual comparisons between the predicted and observed values provide perceivable validation, reinforcing the numerical analysis.
* The focus on core assumptions (e.g., neglecting air resistance) kept the analysis focused, ensuring clarity in isolating gravitational effects.

# Limitations of the Study:

* The manual estimation of y-values within the image introduced potential inaccuracies in the observed data, impacting the validity of model comparisons.
* Some assumptions made in the study, through an ignorance of air resistance and 3D motion, restrict the real-world applicability of the solution despite easing computational complexity.
* The acute alignment of the cubic model indicates possible overfitting, with its intricacy potentially amplifying minor data inconsistencies rather than reflecting true physical behavior.

# Conclusion:

This study developed and evaluated three mathematical models with the goal of accurately simulating the trajectory of a water stream for a computer animation studio. The manually derived quadratic model provided a basic approximation of projectile motion but exhibited significant deviations from observed data due to its rigid symmetry and fixed intercepts. The quadratic regression model improved accuracy by leveraging statistical fitting, reducing residuals by approximately 30%. However, the cubic regression model achieved the closest alignment with the observed trajectory, with residuals 80% lower than the quadratic regression, demonstrating its ability to capture subtle asymmetries and variations in the water stream’s path.

While the cubic model’s flexibility makes it the most suitable choice for replicating real-world nuances, its complexity raises concerns about overfitting to minor data inconsistencies. The study’s assumptions—such as neglecting air resistance and 3D effects—simplified analysis but limit direct real-world applicability. For the animation studio, the cubic regression offers a robust proof of concept for a physics engine, balancing accuracy and computational feasibility. Future work could incorporate additional physical factors (e.g., fluid dynamics) and validate models with higher-precision data to enhance realism. Overall, this project underscores the value of polynomial regression in bridging mathematical abstraction with dynamic visual simulations.

# Reference List:

# Appendix:

### Raw Excel data/calculations

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| x | y (observed value) | y (predicted value) | Residual | y (quadratic value) | Residual (quadratic) | y (cubic value) | Residual (Cubic) |
| 0 | 0 | 0 | 0 | -0.5112 | 0.5112 | 0.1475 | 0.1475 |
| 0.5 | 1.75 | 1.9 | 0.15 | 1.449425 | 0.300575 | 1.712925 | 0.037075 |
| 1 | 3.25 | 3.6 | 0.35 | 3.21 | 0.04 | 3.1823 | 0.0677 |
| 1.5 | 4.65 | 5.1 | 0.45 | 4.770525 | 0.120525 | 4.544075 | 0.105925 |
| 2 | 5.85 | 6.4 | 0.55 | 6.131 | 0.281 | 5.7867 | 0.0633 |
| 2.5 | 6.95 | 7.5 | 0.55 | 7.291425 | 0.341425 | 6.898625 | 0.051375 |
| 3 | 7.85 | 8.4 | 0.55 | 8.2518 | 0.4018 | 7.8683 | 0.0183 |
| 3.5 | 8.65 | 9.1 | 0.45 | 9.012125 | 0.362125 | 8.684175 | 0.034175 |
| 4 | 9.3 | 9.6 | 0.3 | 9.5724 | 0.2724 | 9.3347 | 0.0347 |
| 4.5 | 9.7 | 9.9 | 0.2 | 9.932625 | 0.232625 | 9.808325 | 0.108325 |
| 5 | 10 | 10 | 0 | 10.0928 | 0.0928 | 10.0935 | 0.0935 |
| 5.5 | 10.15 | 9.9 | 0.25 | 10.052925 | 0.097075 | 10.178675 | 0.028675 |
| 6 | 10 | 9.6 | 0.4 | 9.813 | 0.187 | 10.0523 | 0.0523 |
| 6.5 | 9.7 | 9.1 | 0.6 | 9.373025 | 0.326975 | 9.702825 | 0.002825 |
| 7 | 9.2 | 8.4 | 0.8 | 8.733 | 0.467 | 9.1187 | 0.0813 |
| 7.5 | 8.4 | 7.5 | 0.9 | 7.892925 | 0.507075 | 8.288375 | 0.111625 |
| 8 | 7.2 | 6.4 | 0.8 | 6.8528 | 0.3472 | 7.2003 | 0.0003 |
| 8.5 | 5.9 | 5.1 | 0.8 | 5.612625 | 0.287375 | 5.842925 | 0.057075 |
| 9 | 4.2 | 3.6 | 0.6 | 4.1724 | 0.0276 | 4.2047 | 0.0047 |
| 9.5 | 2.2 | 1.9 | 0.3 | 2.532125 | 0.332125 | 2.274075 | 0.074075 |
| 10 | 0 | 0 | 0 | 0.6918 | 0.6918 | 0.0395 | 0.0395 |
| Sum |  |  | 9 |  | 6.2277 |  | 1.21425 |
| Avg |  |  | 0.428571429 |  | 0.296557143 |  | 0.057821429 |